

Week 13

Recall:

The heat equation is the GF of $\int |\nabla u|^2$ in L^2 and of the entropy, $\int \rho \log \rho$, in Wasserstein.

Define given $z > 0$, $\rho_k^z dx$ is the min in $\mathcal{P}(\Omega)$ of $\rho \mapsto \frac{W_2^2(\rho, \rho_{k-1}^z)}{2z} + \int \rho \log \rho$

[we saw a lemma on existence, absol. continuity, and does not concentrate on $\partial\Omega$]

$$\rho^z(t, x) = \rho_k^z$$

Theorem (Jordan, Kinderlehrer, Otto; 1998) (JKO scheme)

There exists $\rho \in L^1_{loc}([0, +\infty) \times \Omega)$ such that

$\rho^z \rightarrow \rho$ up to a subsequence and ρ is a solution to the heat equation starting from ρ_0 .

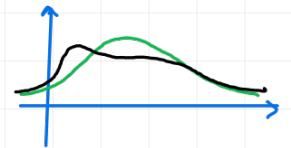
Lemma: $\forall \xi \in C^0(\Omega; \mathbb{R}^d)$ tangent to $\partial\Omega$, $\begin{cases} T_{k+1} \text{ is the OT map from } \rho_k^z \text{ to } \rho_{k+1}^z. \end{cases}$

$$\int_{\Omega} \rho_{k+1}^z \operatorname{div} \xi \, dx = \frac{1}{2} \int_{\Omega} \langle \xi \circ T_{k+1}, T_{k+1} - x \rangle \rho_k^z \, dx$$

Proof:
$$\frac{W_2^2(\rho_{k+1}^z, \rho_k^z)}{2z} + \int \rho_{k+1}^z \log \rho_{k+1}^z \leq \underbrace{\frac{W_2^2(\rho_{\varepsilon}, \rho_k^z)}{2z}}_{(I)} + \underbrace{\int \rho_{\varepsilon} \log \rho_{\varepsilon}}_{(II)} \quad (*)$$

How to define a competitor ρ_{ε} ?

① Outer variation: $\rho_{k+1}^z + \varepsilon \rho_0(x)$, with $\int \rho_0 = 0$



② Inner variation: let ξ as in the statement, $\phi_{\varepsilon}(x) = x + \varepsilon \xi(x)$, $\rho_{\varepsilon} = \phi_{\varepsilon} \# \rho_{k+1}^z$.

③ Inner variation (better done): let ϕ_{ε} be the flow at time ε of ξ :

$$\begin{cases} \partial_t \phi(t, x) = \xi(\phi(t, x)) & (\text{it is tangent to the boundary}). \\ \phi(0, x) = x \end{cases}$$

$$g(T(x)) \det \nabla T(x) = f(x)$$

$$\rho_{\varepsilon} = \phi_{\varepsilon} \# \rho_{k+1}^z \Leftrightarrow \rho_{k+1}^z(x) = \rho_{\varepsilon}(\phi_{\varepsilon}(x)) \det \nabla \phi_{\varepsilon}(x)$$

$$\phi_{\varepsilon}(x) = x + \varepsilon \xi(x) + o(\varepsilon)$$

$$\nabla \phi_{\varepsilon}(x) = \operatorname{Id} + \varepsilon \nabla \xi(x) + o(\varepsilon) \Rightarrow \det \nabla \phi_{\varepsilon}(x) = 1 + \varepsilon \operatorname{div} \xi + o(\varepsilon)$$

$$\hookrightarrow \begin{pmatrix} 1 + \varepsilon \partial_1 \xi & \varepsilon \partial_2 \xi \\ \varepsilon \partial_1 \xi & 1 + \varepsilon \partial_2 \xi \end{pmatrix}$$

we do ③ and not ② because we need $\phi_{\varepsilon} \# \rho_{k+1}^z$ to still be supported in Ω . Otherwise, it is the same.

$$\begin{aligned}
\int \rho_\varepsilon(y) \log \rho_\varepsilon(y) dy &= \int \rho_\varepsilon(\phi_\varepsilon(x)) \log \rho_\varepsilon(\phi_\varepsilon(x)) \det \nabla \phi_\varepsilon(x) dx \\
&= \int \rho_{k+1}^2 (\log \rho_{k+1}^2 - \log \det \nabla \phi_\varepsilon(x)) dx \\
&= \int \rho_{k+1}^2 \log \rho_{k+1}^2 - \varepsilon \int \rho_{k+1}^2 \operatorname{div} \xi dx + o(\varepsilon).
\end{aligned}$$

Let T_{k+1} be the OT map. A transport map between ρ_k^2 and ρ_ε is $\phi_\varepsilon \circ T_{k+1}$.

$$\begin{aligned}
\frac{W_2^2(\rho_\varepsilon, \rho_k^2)}{22} &\leq \frac{1}{22} \int \underbrace{|\phi_\varepsilon \circ T_{k+1}(x) - x|^2}_{T_{k+1}(x) + \varepsilon \xi(T_{k+1}(x))} \rho_k^2(x) dx \\
&\leq \frac{1}{22} \int \left[|T_{k+1}(x) - x|^2 + 2\varepsilon \langle T_{k+1}(x) - x, \xi(T_{k+1}(x)) \rangle \right] \rho_k^2(x) dx + o(\varepsilon) \\
&= \frac{1}{22} W_2^2(\rho_{k+1}^2, \rho_k^2) + \frac{2\varepsilon}{22} \int \langle T_{k+1}(x) - x, \xi(T_{k+1}(x)) \rangle \rho_k^2(x) dx + o(\varepsilon)
\end{aligned}$$

Plugging these in (*):

$$0 \leq -\varepsilon \int \rho_{k+1}^2 \operatorname{div} \xi dx + \frac{\varepsilon}{2} \int \langle T_{k+1}(x) - x, \xi(T_{k+1}(x)) \rangle \rho_k^2(x) dx + o(\varepsilon).$$

Divide by ε , let $\varepsilon \downarrow 0$, and change ξ by $\xi - \xi$ Still divergent to $\partial\Omega$ to get the desired result.

Def: $\rho \in L^1_{loc}$ satisfies the heat equation (HE) distributionally starting from

ρ_0 if $\forall \psi \in C_c^\infty(\Omega), \xi \in C_c^\infty([0, \infty))$ we have:

$$\xi(0) \int_\Omega \psi \rho_0 dx + \int_0^\infty \int_\Omega \rho_t (\psi(x) \partial_t \xi(t) + \Delta \psi(x) \xi(t)) dx dt = 0$$

• with Neumann boundary conditions if $\psi \in C^\infty(\Omega)$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$ instead.
 → definition to avoid regularity issues ($\rho \in L^1_{loc}$).

Remark: it is actually equivalent to being a classical solution.

Remark 2: Why is it a reasonable definition?

If $\rho \in C^2$ solves $\begin{cases} \partial_t \rho - \Delta \rho = 0 & \text{in } \Omega \times [0, \infty) \\ \frac{\partial \rho}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$ then it is satisfied. (A) (B)

Indeed, test the PDE with $\psi(x) \xi(t)$: $\int_0^\infty \int_\Omega (\partial_t \rho \psi(x) \xi(t) - \Delta \rho \psi(x) \xi(t)) = 0$

$$(A) = - \int_\Omega \psi(x) \int_0^\infty \rho \partial_t \xi(t) - \int_\Omega \psi(x) \xi(0) \rho(0, x)$$

$$(B) \text{ We know: } \nabla \rho \cdot \nabla \psi + \Delta \rho \psi = \operatorname{div}(\nabla \rho \cdot \psi) \rightarrow \text{integrated is } \int_{\partial\Omega} \frac{\partial}{\partial \nu} \rho \psi = 0$$

$$\text{thus } (B) = \int_0^\infty \int_\Omega \nabla \rho \cdot \nabla \psi \xi(t) = - \int_0^\infty \int_\Omega \Delta \psi \rho$$

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu} \psi \rho = 0 \leftarrow \operatorname{div}(\nabla \psi \rho) = \nabla \psi \cdot \nabla \rho + \Delta \psi \rho$$

Proof of the main theorem (JKO): ρ_{k-1}^2 is a competitor

$$\frac{\omega_2^2(\rho_k^2, \rho_{k-1}^2)}{2} + \int \rho_k^2 \log \rho_k^2 \leq \int \rho_{k-1}^2 \log \rho_{k-1}^2 \xrightarrow{\text{sum in } k}$$

$$\sum_k \rightarrow \underbrace{\sum_{k=1}^{k_0} \frac{\omega_2^2(\rho_k^2, \rho_{k-1}^2)}{2}}_{\geq 0} + \int \rho_{k_0}^2 \log \rho_{k_0}^2 \leq \int \rho_0 \log \rho_0 \quad (+)$$

measures cannot concentrate on $\partial\Omega$ or escape uniformly in t

$$\Rightarrow \text{entropy decreasing in } k: \int \rho^2(t, x) \log \rho^2(t, x) \leq \int \rho_0 \log \rho_0 \quad \forall t.$$

$$\text{Since } \int \rho^2(t, x) dx = 1 \rightarrow \int_0^T \int_{\Omega} \rho^2(t, x) = T \quad (**); \text{ and up to a}$$

$$\text{subsequence, } \rho^2 \rightharpoonup \rho \in L^1 \text{ in } L^1([0, T] \times \Omega) \quad \forall T > 0,$$

why? same as last week, based on (+).

$$\text{and by } (**) \text{ in any time interval, } \int \rho = 1 \text{ for a.e. } t \in [0, T].$$

By today's lemma, applied to $\xi = \nabla \psi$, $\psi \in C^\infty(\Omega)$, $\partial_\nu \psi = 0$ on $\partial\Omega$,

$$- \int \Delta \psi \rho_k^2 dx + \frac{1}{2} \int_{\Omega} \langle \nabla \psi \circ T_k, T_k - x \rangle \rho_{k-1}^2 dx = 0$$

$$\hookrightarrow \text{Taylor expansion: } | -\psi(T_k) + \psi(x) - \langle \nabla \psi(T_k), x - T_k \rangle | \leq \|D^2 \psi\|_{L^\infty} |x - T_k|^2.$$

$$\text{So: } \left| - \int_{\Omega} \Delta \psi \rho_k^2 dx - \frac{1}{2} \int_{\Omega} \psi \rho_{k-1}^2 + \frac{1}{2} \int_{\Omega} \psi \rho_k^2 \right| \leq \|D^2 \psi\|_{L^\infty} \frac{\omega_2^2(\rho_{k-1}^2, \rho_k^2)}{2}.$$

Now take $\xi \in C_c^\infty([0, \infty))$ and multiply against $2\xi((k-1)2)$,

$$\begin{aligned} & \left| -2 \int_{\Omega} \Delta \psi \rho_k^2 \xi((k-1)2) - \int_{\Omega} \psi(x) \rho^2((k-1)2, x) \xi((k-1)2) + \int_{\Omega} \psi(x) \rho^2(k2, x) \xi((k-1)2) \right| \\ & \leq \underbrace{\|D^2 \psi\|_{L^\infty} \|\xi\|_{L^\infty}}_{\leq C} \omega_2^2(\rho^2(k2, \cdot), \rho^2((k-1)2, \cdot)). \end{aligned}$$

Sum over k :

II

$$\begin{aligned} & \left| -2 \sum_{k=1}^{\infty} \int_{\Omega} \Delta \psi \rho^2(k2, x) \xi((k-1)2) - \xi(0) \int_{\Omega} \psi(x) \rho_0(x) dx \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \int_{\Omega} \psi(x) \rho^2(k2, x) \xi((k-1)2) - \sum_{k=1}^{\infty} \int_{\Omega} \psi(x) \rho^2(k2, x) \xi(k2) \right| \leq \quad (X) \\ & \leq C \sum_{k=1}^{\infty} \omega_2^2(\rho^2(k2, \cdot), \rho^2((k-1)2, \cdot)) \leq C2 \rightarrow 0 \text{ as } 2 \rightarrow 0 \\ & \quad \hookrightarrow (+) \end{aligned}$$

For I:

$$= \sum_{k=1}^{\infty} \int_{\Omega} \psi(x) \rho^2(t, x) \left[\overbrace{\xi((k-1)\tau) - \xi(k\tau)}^{\substack{t \in ((k-1)\tau, k\tau) \\ \int_{(k-1)\tau}^{k\tau} \partial_t \xi}} \right] = - \int_0^{\infty} \int_{\Omega} \psi(x) \rho^2(t, x) \partial_t \xi(t) dx dt$$

as $\tau \rightarrow 0$, since $\rho^2 \rightarrow \rho$ weakly in L^1 this term goes to

$$\rightarrow \int_0^{\infty} \int_{\Omega} \psi(x) \rho(t, x) \partial_t \xi(t) dx dt.$$

$$\text{For II: } \tau \xi((k-1)\tau) = \int_{(k-1)\tau}^{k\tau} \xi((k-1)\tau) dt \stackrel{\|\nabla \xi\|_{L^\infty} \leq C}{=} \int_{(k-1)\tau}^{k\tau} \xi(t) dt + O(\tau)$$

$$\text{II} = - \int_0^{\infty} \int_{\Omega} \xi(t) \Delta \psi \rho^2(t, x) dx dt + O(\tau)$$

$$\text{as } \tau \rightarrow 0, \text{ this term goes to } \rightarrow - \int_0^{\infty} \int_{\Omega} \xi(t) \Delta \psi \rho(t, x) dx dt$$

Sending $\tau \downarrow 0$ in (X) we get:

$$\xi(0) \int_{\Omega} \psi \rho_0 dx + \int_0^{\infty} \int_{\Omega} [\xi(t) \Delta \psi(x) \rho(t, x) + \partial_t \xi(t) \psi(x) \rho(t, x)] = 0.$$

This is the definition of distributional solution to the heat equation. $\square$